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# PLASTIC BUCKLING OF PLATES AND SHELLS UNDER BIAXIAL LOADING

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✓ NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# PLASTIC BUCKLING OF PLATES AND SHELLS UNDER BIAXIAL LOADING

By James P. Peterson  
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## SUMMARY

A method of calculating the plastic buckling load of plates and shells under biaxial loading is described. This method uses available elastic buckling equations in making the plastic buckling calculation by substituting wall stiffnesses derived from a plastic buckling theory for the elastic orthotropic wall stiffnesses associated with the elastic buckling equations.

This method permits the calculation of the buckling load of many structures of current interest which cannot be handled by available methods. Buckling loads of cylinders and plates of conventional, sandwich, and laminated construction are easily calculated. The cylinders and plates may have stiffening members which may be eccentric (one-sided) and which may be made of a different material than the skin. For many structures, the effect of transverse shear can be taken into account in the buckling calculations.

## INTRODUCTION

Many aerospace structures buckle at stresses greater than the proportional-limit stress of the material used in the fabrication of the structure. Such structures are normally designed with the use of buckling equations for elastic structures in conjunction with estimates of the deleterious effects of plasticity on the computed buckling load. The estimates are derived from experience with available plastic buckling analyses and from the plastic buckling tests of simple structures that are reported in the literature.

Solutions to plastic buckling problems are generally limited to simple plates and shells subjected to rather simple loadings (for instance, refs. 1 to 3). Notable exceptions are reference 4, which treats clad plates; reference 5, which treats sandwich plates; reference 6, which treats a stiffened cylinder; references 7 and 8, which treat a conventional cylinder under hydrostatic loading; reference 9, which treats a stiffened cylinder under hydrostatic loading; and reference 10, which treats a rib-core sandwich cylinder under hydrostatic loading.

However, solutions analogous to those of the corresponding plastic buckling problems are available for elastic orthotropic plates and shells subjected to any biaxial loading. These solutions can be used to calculate the plastic buckling load of plates and shells of rather complex construction. The purpose of the present paper is to indicate this analogy between plastic buckling of plates and shells and elastic buckling of orthotropic plates and shells and thereby provide a ready means of calculating the plastic buckling load of many structures of interest for which solutions are not otherwise available. In addition, several examples are given to illustrate the utility of the analogy in computing the buckling load of typical structures which buckle at stresses greater than the proportional-limit stress.

## SYMBOLS

The units used for physical quantities defined in this paper are given both in the U.S. Customary Units and in the International System of Units (SI). Factors relating the two systems are given in reference 11 and those used in the present paper are given in appendix A.

$A, B, C$	constants defined by equations (D2)
$A_s, A_r$	area of stringer and rib or ring, respectively
$b$	width of flat or curved plate, semicircumference of cylinder
$b_r$	distance between ribs or rings
$b_s$	distance between stringers
$B$	extensional stiffness parameter (see eqs. (2) and (4))
$d$	distance from centroid of laminated plate to centroid of lamina of plate
$D$	bending stiffness parameter (see eqs. (2) and (4))
$\bar{D}_c$	bending stiffness parameter of core element of sandwich (see eq. (C6))
$D_e$	elastic value of bending stiffness parameter
$\bar{D}_f$	bending stiffness parameter of face sheet of sandwich (see eq. (C5))

$D_Q$	transverse shear stiffness of plate
$D_{Q_x}, D_{Q_y}$	transverse shear stiffness of plate in x- and y-direction, respectively
$D_x, D_y$	bending stiffness of plate in x- and y-direction, respectively
$\bar{D}_x, \bar{D}_y$	bending stiffness parameters defined by equations (1)
$D_{xy}$	twisting stiffness of plate
$D_{12} = 2D_{xy} + \mu_x \bar{D}_y + \mu_y \bar{D}_x$	
$E$	Young's modulus
$E_c$	modulus of core element
$\bar{E}_c$	extensional stiffness parameter of core (see eq. (C4))
$E_s, E_r$	modulus of stringer and rib or ring, respectively
$E_{sec}$	secant modulus
$E_{tan}$	tangent modulus
$E_x, E_y$	extensional stiffness of plate in x- and y-direction, respectively
$\bar{E}_x, \bar{E}_y$	extensional stiffness parameters defined by equations (1)
$E_{12} = \frac{E_x E_y}{G_{xy}} - \mu'_x E_y - \mu'_y E_x$	
$G_s, G_r$	shear modulus of stringer and rib or ring, respectively
$G_{sec} = \frac{E_{sec}}{2(1 + \mu)}$	
$G_{xy}$	in-plane shear stiffness of plate
$G_{xz}, G_{yz}$	shear modulus of core of sandwich in xz- and yz-plane, respectively
$h$	depth of sandwich measured between centroids of face sheets

$I_s, I_r$	moment of inertia about centroid of symmetrically stiffened plate of stringer and rib or ring, respectively
$J_s, J_r$	torsion constant of stringer and rib or ring, respectively
$k_x$	buckling coefficient
$K$	constant used in defining stress-strain curve
$K_\mu$	plasticity factor defined in equations (2)
$l$	length of plate or cylinder
$M_x, M_y$	bending moment in plate in x- and y-direction, respectively
$M_{xy}$	twisting moment in plate
$m, n$	number of buckles (half waves) in buckle pattern in x- and y-direction, respectively (For cylinder, $n$ denotes number of buckles in semicircumference.)
$N_x, N_y$	normal forces in plate in x- and y-direction, respectively
$N_{xy}$	shearing force in plate
$\bar{N}_x, \bar{N}_y$	applied loads in plate or cylinder wall at buckling in x- and y-direction, respectively
$\bar{N}_0$	value of $\bar{N}_x$ for sandwich cylinder when shear stiffness of core is assumed large
$p$	internal pressure
$Q_x, Q_y$	transverse shear forces in plate or cylinder wall in x- and y-direction, respectively
$r$	radius
$t$	skin thickness
$t_c$	thickness of core element of truss-core or web-core sandwich

$t_f$	thickness of face sheet of sandwich
$\bar{t}$	so-called "smeared out" thickness obtained if mass per unit area of wall is divided by density of principal material component of wall
$u, v$	displacement of a point in middle surface of plate or cylinder in x- and y-direction, respectively
$w$	displacement normal to surface of plate or cylinder
$x, y$	axial and chordwise or circumferential direction, respectively
$\alpha$	plasticity factor defined in equations (2)
$\beta$	buckle aspect ratio
$\beta_x, \beta_y, \beta_{xy}$	plasticity factors defined in equations (2)
$\epsilon$	strain
$\epsilon_x, \epsilon_y$	strain in the x- and y-direction, respectively
$\epsilon_1, \epsilon_2, \epsilon_3$	variations in middle surface strain
$\eta$	plasticity factor
$\mu$	Poisson's ratio
$\mu_e$	elastic value of Poisson's ratio
$\mu_x, \mu_y$	Poisson's ratio associated with bending of plate in x- and y-direction, respectively
$\mu'_x, \mu'_y$	Poisson's ratio associated with extension of plate in x- and y-direction, respectively
$\sigma$	stress
$\sigma_i$	stress intensity (see eq. (3))

$\sigma_{cy}$	compressive yield stress corresponding to 0.2-percent offset (permanent set)
$\sigma_{pl}$	stress corresponding to proportional limit of material
$\sigma_s, \sigma_r$	stress in stringer and rib or ring, respectively
$\sigma_x, \sigma_y$	stress in plate in x- and y-direction, respectively
$\varphi$	angle between core element of truss-core or web-core sandwich and face sheet
$\chi_1, \chi_2, \chi_3$	variations in middle-surface curvature
Subscript:	
i	denotes ith lamina, except when used as subscript on $\sigma$

## THEORY

A one-to-one correspondence exists between certain terms in plastic buckling equations for conventional (isotropic, homogeneous) plates and shells under biaxial loading (for instance, refs. 2, 3, 6, 7, 8, and 9) and the corresponding terms in elastic buckling equations for orthotropic plates and shells (for instance, refs. 12 and 13). Although the correspondence is limited to problems in which structures are assumed to buckle under a constant (membrane) state of stress, many problems of current interest are included. The correspondence makes writing the solution to these plastic buckling problems possible simply by redefining stiffness terms in the existing elastic solutions for orthotropic plates and shells. Moreover, the correspondence is not limited to conventional plates and shells but can be applied to sandwich-like plates and shells (refs. 5 and 10) and to laminated plates and shells (ref. 4) such as those constructed of clad sheet materials.

The plates and shells may be stiffened by stringers and ribs or rings which may be either symmetrically disposed about the middle surface of the skin or may be eccentric to the middle surface.

The equivalent orthotropic constants for plastic plates are obtained by comparing corresponding terms from the two theories. For instance, for conventional unstiffened plastic plates and shells subjected to biaxial loading, the equations expressing the variation of forces and moments during buckling in terms of the variations of middle-surface strains and curvatures can be written (ref. 3 or ref. 9) as



$$\begin{aligned}
N_x &= B \left[ (1 - \alpha\beta_x)\epsilon_1 + \mu(1 - \alpha\beta_{xy})\epsilon_2 \right] \\
N_y &= B \left[ \mu(1 - \alpha\beta_{xy})\epsilon_1 + (1 - \alpha\beta_y)\epsilon_2 \right] \\
N_{xy} &= B(1 - \mu)\epsilon_3 \\
M_x &= -D \left[ (1 - \alpha\beta_x)\chi_1 + \mu(1 - \alpha\beta_{xy})\chi_2 \right] \\
M_y &= -D \left[ \mu(1 - \alpha\beta_{xy})\chi_1 + (1 - \alpha\beta_y)\chi_2 \right] \\
M_{xy} &= D(1 - \mu)\chi_3
\end{aligned}$$

These equations have the same form as the corresponding equations from elastic buckling problems of orthotropic plates and shells. Moreover, the correspondence is not limited to these equations but continues to hold throughout the entire buckling problem. Hence, for conventional plates and shells, the equivalent orthotropic constants are identified as

$$\left. \begin{aligned}
\bar{E}_x &= \frac{E_x}{1 - \mu'_x \mu'_y} = B(1 - \alpha\beta_x) \\
\bar{E}_y &= \frac{E_y}{1 - \mu'_x \mu'_y} = B(1 - \alpha\beta_y) \\
G_{xy} &= \frac{B(1 - \mu)}{2} \\
\bar{D}_x &= \frac{D_x}{1 - \mu'_x \mu'_y} = D(1 - \alpha\beta_x) \\
\bar{D}_y &= \frac{D_y}{1 - \mu'_x \mu'_y} = D(1 - \alpha\beta_y) \\
D_{xy} &= D(1 - \mu) \\
\mu_x &= \mu'_x = \mu \frac{1 - \alpha\beta_{xy}}{1 - \alpha\beta_y} \\
\mu_y &= \mu'_y = \mu \frac{1 - \alpha\beta_{xy}}{1 - \alpha\beta_x}
\end{aligned} \right\} \quad (1)$$

where

$$\begin{aligned}
 B &= \frac{E_{\text{sec}} t}{1 - \mu^2} \\
 D &= \frac{E_{\text{sec}} t^3}{12(1 - \mu^2)} \\
 \alpha &= \frac{(2 - \mu)^2}{1 - \mu^2} \frac{1 - \frac{E_{\text{tan}}}{E_{\text{sec}}}}{K_{\mu} \sigma_i^2} \\
 \beta_x &= \frac{1}{4} \left( \sigma_x - \frac{1 - 2\mu}{2 - \mu} \sigma_y \right)^2 \\
 \beta_y &= \frac{1}{4} \left( \sigma_y - \frac{1 - 2\mu}{2 - \mu} \sigma_x \right)^2 \\
 \beta_{xy} &= \frac{1}{4\mu} \left\{ \sigma_x \sigma_y \left[ 1 + \left( \frac{1 - 2\mu}{2 - \mu} \right)^2 \right] - (\sigma_x^2 + \sigma_y^2) \frac{1 - 2\mu}{2 - \mu} \right\} \\
 K_{\mu} &= 1 - \frac{(1 - 2\mu) \left( 1 - \frac{E_{\text{tan}}}{E_{\text{sec}}} \right)}{2(1 - \mu^2)} \left\{ 2\mu - \frac{1}{2\sigma_i^2} \left[ (1 + 2\mu) (\sigma_x^2 + \sigma_y^2) - 2(2 + \mu) \sigma_x \sigma_y \right] \right\}
 \end{aligned} \tag{2}$$

Poisson's ratio  $\mu$  may be taken either as  $\mu = \frac{1}{2}$ , as in references 2 to 5, or as  $\mu = \frac{1}{2} - \left( \frac{1}{2} - \mu_e \right) \frac{E_{\text{sec}}}{E}$ , as in references 6 and 7. For  $\mu = \frac{1}{2}$ ,  $K_{\mu}$  becomes unity and several of the other terms simplify to a lesser extent.

The moduli  $E_{\text{sec}}$  and  $E_{\text{tan}}$  are evaluated at the stress intensity  $\sigma_i$  of the plate at buckling with the use of a conventional stress-strain curve of the plate material. The stress intensity is given by

$$\sigma_i = \sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2} \tag{3}$$

where  $\sigma_x$  and  $\sigma_y$  are the constant membrane stresses in the plate at buckling.

Although equations (1) were written for conventional plates and shells, they can be applied to buckling problems involving sandwich-like plates and shells with isotropic face sheets by a redefinition of the stiffness parameters  $B$  and  $D$  in equations (2). For sandwich-like plates and shells

$$\left. \begin{aligned} B &= \frac{2E_{\text{sec}}t_f}{1 - \mu^2} \\ D &= \frac{E_{\text{sec}}t_f h^2}{2(1 - \mu^2)} + \frac{E_{\text{sec}}t_f^3}{6(1 - \mu^2)} \end{aligned} \right\} \quad (4)$$

where the second term on the right-hand side of the expression for  $D$  is normally neglected in comparison with the first right-hand term, and the stiffness of the core is neglected in comparison with the stiffness of the face sheets. If more accurate buckling calculations in which core stiffness is taken into account are desired, the core can be considered to be a stiffening member such as a stringer or rib. The inclusion of the effect of symmetrically disposed, load-carrying core elements (stiffeners) in the stiffness relationships is discussed in appendix B and expressions for the transverse shear stiffness of web-core and truss-core sandwich plates are given in appendix C.

For symmetrically laminated isotropic plates and shells, equations (1) and (2) apply to each lamina and the plate stiffnesses can be written in terms of the lamina stiffnesses. For laminated plates

$$\left. \begin{aligned} \bar{E}_x &= \sum_i \bar{E}_{x_i} \\ \bar{E}_y &= \sum_i \bar{E}_{y_i} \\ G_{xy} &= \sum_i G_{xy_i} \\ \bar{D}_x &= \sum_i (\bar{D}_{x_i} + \bar{E}_{x_i} d_i^2) \\ \bar{D}_y &= \sum_i (\bar{D}_{y_i} + \bar{E}_{y_i} d_i^2) \\ D_{xy} &= \sum_i (D_{xy_i} + 2G_{xy_i} d_i^2) \\ \mu'_x &= \frac{\sum_i \mu'_{x_i} \bar{E}_{y_i}}{\bar{E}_y} \end{aligned} \right\}$$

(Equations continued on next page)

$$\left. \begin{aligned} \mu'_y &= \frac{\sum_i \mu'_{y_i} \bar{E}_{x_i}}{\bar{E}_x} \\ \mu_x &= \frac{\sum_i \mu_{x_i} (\bar{D}_{y_i} + \bar{E}_{y_i} d_i^2)}{\bar{D}_y} \\ \mu_y &= \frac{\sum_i \mu_{y_i} (\bar{D}_{x_i} + \bar{E}_{x_i} d_i^2)}{\bar{D}_x} \end{aligned} \right\} \quad (5)$$

where the lamina constants are given by equations (1) and (2). Application of equations (5) is simplified somewhat if Poisson's ratio  $\mu$  for each lamina is taken equal to  $1/2$ , as is normally done.

The preceding equations for conventional, sandwich, and laminated plates apply both to stiffened and to unstiffened plates. If the plates are stiffened by stringers, ribs, or rings, the appropriate modulus for extension and bending of the stiffening members in the buckling equations is  $E_{tan}$  and the appropriate shearing modulus in expressions for torsional stiffness of the members is  $G_{sec}$  where  $G_{sec} = \frac{E_{sec}}{2(1 + \mu)}$ . Both  $E_{tan}$  and  $E_{sec}$  for the members are evaluated at the stress in the particular stiffening member at buckling which may be different from the stress intensity of the plate or from the stress in the plate in the direction of the stiffening member.

The stresses in the stiffening members and in the stiffened plate are given by the relationships

$$\bar{N}_x = B(\epsilon_x + \mu \epsilon_y) + \frac{E_s A_s}{b_s} \epsilon_x$$

$$\bar{N}_y = B(\epsilon_y + \mu \epsilon_x) + \frac{E_r A_r}{b_r} \epsilon_y$$

$$\sigma_x = \frac{E_{sec}}{1 - \mu^2} (\epsilon_x + \mu \epsilon_y)$$

$$\sigma_y = \frac{E_{sec}}{1 - \mu^2} (\epsilon_y + \mu \epsilon_x)$$

$$\sigma_s = E_s \epsilon_x$$

$$\sigma_r = E_r \epsilon_y$$

where  $\bar{N}_x$  and  $\bar{N}_y$  are the applied loads,  $E_s$  and  $E_r$  denote the secant moduli of the respective members at the strain associated with the member ( $\epsilon_x$  or  $\epsilon_y$ ) and  $B$  for a laminated plate is understood to be the summation of the  $B$  values for the individual laminae. For conventional or sandwich plates, it is convenient to assume values for  $\sigma_x$  and  $\sigma_y$  and compute  $\epsilon_x$  and  $\epsilon_y$ . Then the remaining quantities needed to determine the applied loads and the stiffness constants required to obtain the buckling load are computed. This scheme also works for laminated plates if Poisson's ratio is taken equal to  $1/2$ , as is normally done. If Poisson's ratio is different in different laminae, it may be more convenient to assume a strain distribution defined by  $\epsilon_x$  and  $\epsilon_y$ , determine  $\sigma_x$  and  $\sigma_y$  for each lamina by trial and error with the use of equation (3), and then compute the other needed quantities.

The equations presented in this section and an appropriate buckling equation for elastic orthotropic plates or shells (for instance, ref. 13 or appendix D) are all that is required for calculation of the plastic buckling load of many plate and shell structures. The plasticity factors and buckling equations of references 2 to 10 can be readily derived with the use of the presented equations. In addition, solutions to buckling problems for many structures not covered by published plastic buckling equations can be readily obtained. In the following section, some published plasticity factors and plastic buckling equations are discussed to illustrate the use of the equations presented and some numerical examples are given for selected structures to illustrate the effect of plasticity on buckling of contemporary structures.

## APPLICATIONS

The application of the equations of the previous section normally entails a trial-and-error solution to the buckling problem. A stress level in the plate or shell is assumed and the necessary stiffness constants (which are functions of the stress level) are computed. These constants are then substituted into a stability equation for orthotropic plates or shells in order to obtain a computed buckling stress. If the computed buckling stress does not agree with the assumed stress, a new stress is assumed and the process is repeated. This procedure is continued until the computed stress agrees with the assumed stress.

In simpler buckling problems in which closed-form solutions of the buckling load in terms of pertinent orthotropic plate stiffnesses are feasible, plasticity factors which can be used to correct calculations made with the use of elastic buckling equations for isotropic plates and shells can be obtained. The need for a trial-and-error solution is thereby circumvented. Some problems of this type, for which solutions already exist in

the literature, are discussed first to demonstrate the use of the proposed method on simpler structures. Then results of numerical examples on more complicated structures are given to show the significance of plasticity in the design of contemporary structures. Trial-and-error solutions are employed to obtain the latter results.

### Closed-Form Solutions

The applications discussed in this section make use of the buckling equation for orthotropic plates and cylinders presented in appendix D (eq. (D1)). For a flat plate with negligible transverse shearing deformations, equation (D1) reduces to

$$k_x \beta^2 = 1 + \frac{D_{12}}{\bar{D}_x} \beta^2 + \frac{\bar{D}_y}{\bar{D}_x} \beta^4 \quad (6)$$

where

$$D_{12} = 2\mu_y \bar{D}_x + 2D_{xy}$$

$$k_x = \frac{\bar{N}_x b^2}{\pi^2 \bar{D}_x}$$

Hence for a wide plate-column  $\left(\frac{l}{b} \ll 1\right)$

$$\bar{N}_x = \frac{\pi^2}{l^2} \bar{D}_x = \frac{\pi^2}{l^2} \eta D_e \quad (7)$$

and, with the use of equations (1) and (2),

$$\eta = \frac{\bar{D}_x}{D_e} = \frac{1 - \mu_e^2}{1 - \mu^2} \frac{E_{sec}}{E} (1 - \alpha \beta_x) \quad (8)$$

For  $\mu = \mu_e = \frac{1}{2}$  equation (8) reduces to

$$\eta = \frac{E_{sec}}{E} \left( \frac{1}{4} + \frac{3}{4} \frac{E_{tan}}{E_{sec}} \right) \quad (9)$$

the value given in reference 2. For a clad plate  $\left(\frac{l}{b} \ll 1\right)$ , the plasticity factor  $\eta$  is again given by  $\eta = \frac{\bar{D}_x}{D_e}$ . For  $\mu = \mu_e = \frac{1}{2}$  and for a cladding thickness given by "a" times the core thickness, equations (1), (2), and (5) yield

$$\eta = \frac{E_{sec} \left( \frac{1}{4} + \frac{3}{4} \frac{E_{tan}}{E_{sec}} \right) + g \tilde{E}_{sec} \left( \frac{1}{4} + \frac{3}{4} \frac{\tilde{E}_{tan}}{\tilde{E}_{sec}} \right)}{E(1 + g)} \quad (10)$$

where  $E_{\text{sec}}$  and  $E_{\text{tan}}$  with the overhead tilde denote the moduli of the cladding,  $E_{\text{sec}}$  and  $E_{\text{tan}}$  without the tilde denote the moduli of the core, and the quantity  $\frac{g}{1+g}$  denotes the contribution of the cladding to the moment of inertia of the clad plate. The quantity  $g$  is given by

$$g = 8a^3 + 12a^2 + 6a$$

Equation (10) was first presented in reference 4.

The other plasticity factors in references 2 and 4 could be similarly rederived. For example, for simply supported long plates, equation (6) after minimization with respect to  $\beta$  becomes

$$\bar{N}_x = \frac{\pi^2}{b^2} \left( 2\sqrt{\bar{D}_x \bar{D}_y} + D_{12} \right) = \frac{\pi^2}{b^2} (4\eta D_e) \quad (11)$$

and the plasticity factor becomes

$$\eta = \frac{\sqrt{\bar{D}_x \bar{D}_y} + \frac{D_{12}}{2}}{2D_e} \quad (12)$$

When the stiffnesses for isotropic and laminated (clad) plates with  $\mu = \mu_e = \frac{1}{2}$  are substituted into equation (12), the corresponding plasticity factors of references 2 and 4, respectively, are obtained. When the stiffnesses for isotropic sandwich plates (eqs. (4)) are substituted into equation (12), a plasticity factor for sandwich plates which corresponds to the factor implied by equation (B10) of reference 5 for the shear-stiffness parameter  $s$  of that reference set equal to zero (large transverse shear stiffness) is obtained. Moreover, when the stiffnesses for sandwich plates with  $\mu = \mu_e = \frac{1}{2}$  and with  $D_{Q_x} = D_{Q_y} = D_Q$  are substituted into equation (D1), equation (B10) of reference 5 is obtained. Hence the presented equations provide an easy means of accounting for plasticity in simple buckling problems of plates and shells.

### Numerical Examples

The three numerical examples given in this section include a ring-and-stringer-stiffened cylinder under axial load and internal pressure, a sandwich cylinder under axial load, and a ring-stiffened cylinder subjected to biaxial compressive loads. Both stiffened cylinders have the stiffening elements attached to the inside surface of the skin.

Calculations for the numerical examples were made with the use of a digital computer and analytical expressions for Poisson's ratio and for the stress-strain curves. Poisson's ratio was taken as

$$\mu = \frac{1}{2} - \left( \frac{1}{2} - \mu_e \right) \frac{E_{\text{sec}}}{E}$$

and the stress-strain curves were defined by

$$\epsilon = \frac{\sigma}{E} + 0.002 \left( \frac{\sigma - \sigma_{pl}}{\sigma_{cy} - \sigma_{pl}} \right)^K \quad (\sigma > \sigma_{pl})$$

The stress-strain curves used are given in figure 1. Material constants used in the construction of the curves are

	Material constants for -		
	2219-T87 aluminum alloy	7075-T6 aluminum alloy	HY-100 steel
E . . . .	10 500 ksi (72.4 GN/m <sup>2</sup> )	10 500 ksi (72.4 GN/m <sup>2</sup> )	30 000 ksi (207 GN/m <sup>2</sup> )
$\sigma_{pl}$ . . .	45 ksi (310 MN/m <sup>2</sup> )	55 ksi (379 MN/m <sup>2</sup> )	85 ksi (586 MN/m <sup>2</sup> )
$\sigma_{cy}$ . . .	60 ksi (414 MN/m <sup>2</sup> )	72 ksi (496 MN/m <sup>2</sup> )	100 ksi (690 MN/m <sup>2</sup> )
K . . . .	3.0	2.2	3.5
$\mu_e$ . . .	0.32	0.32	0.30

Ring-and-stringer-stiffened cylinder.- The stiffening members for the ring-and-stringer-stiffened cylinder analyzed are shown in figure 2. Other pertinent dimensions of the cylinder are wall thickness, 0.120 inch (3.05 mm); ring spacing, 20.0 inches (5.08 cm); stringer spacing, 3.0 inches (7.62 cm); radius, 200 inches (5.08 m); and cylinder length, 200 inches (5.08 m). The structure is representative of those which might be considered for the wall of a fuel or oxidizer tank of a large launch vehicle. The wall and stiffener material chosen (2219-T87 aluminum alloy) is considered to be weldable. The stress-strain curve of this material is given in figure 1.

Calculated results of the pressurized ring-and-stringer-stiffened cylinder for axial load and internal pressure applied simultaneously and proportionately are given in figure 3. The results were obtained with the use of a buckling equation (eq. (15) of ref. 13) for stiffened orthotropic cylinders. Several interesting phenomena are illustrated in figure 3. The maximum load for the structure  $\frac{\bar{N}_x}{r} + \frac{p}{2}$  occurs at an internal pressure of approximately 35 psi (241 kN/m<sup>2</sup>). This load is associated with considerable plasticity; that is, the computed plastic buckling load carried by compressive stresses in the wall is approximately 68 percent of the computed elastic buckling load. At this load the average axial stress in the skin and stringers is less than the proportional-limit stress of the skin and stringer material under uniaxial load but, of course, the stress intensity in the cylinder wall is greater than the proportional-limit stress. This stress



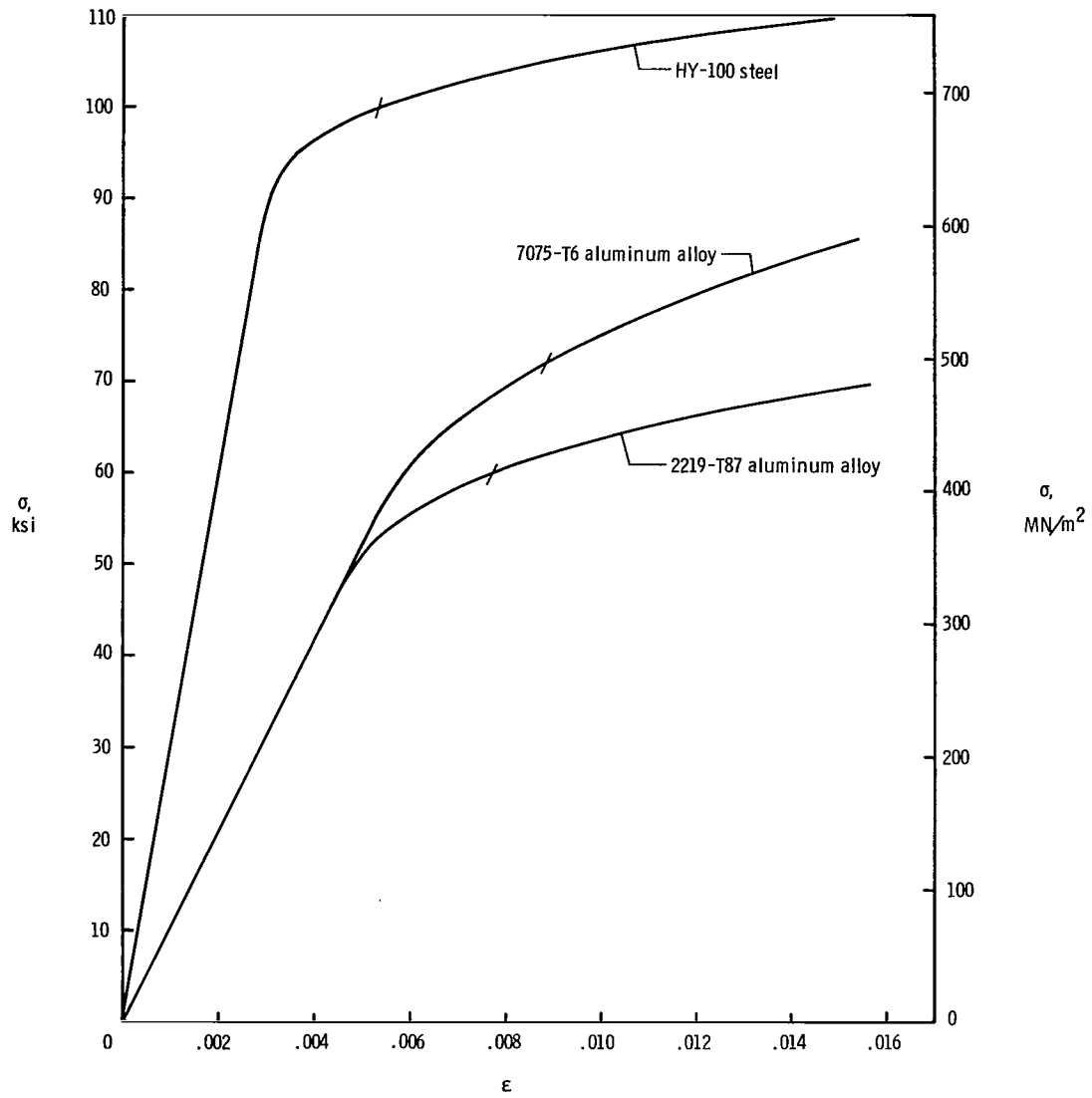


Figure 1.- Stress-strain curves of materials used in numerical examples.

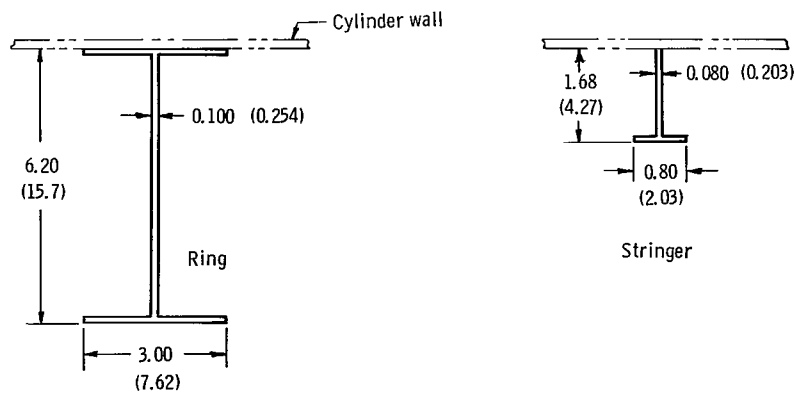


Figure 2.- Stiffening members for ring-and-stringer-stiffened cylinder. All dimensions are given in inches and parenthetically in centimeters.

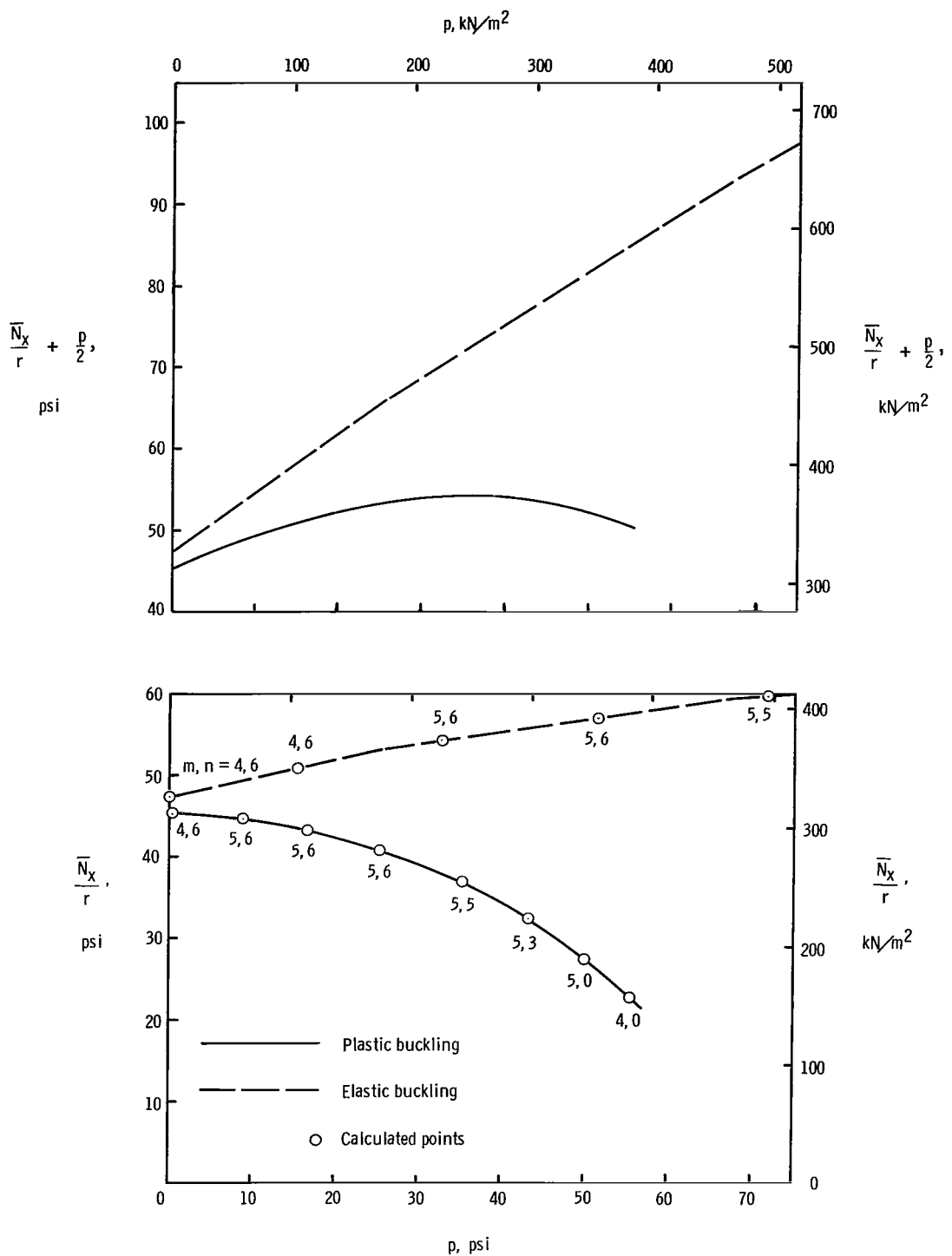


Figure 3.- Results of calculations to determine the buckling strength of the pressurized ring-and-stringer-stiffened cylinder under axial compression.

distribution indicates the difficulty that would be encountered in improvising a plasticity factor for this case as is sometimes done for simpler structures under simpler loadings. Moreover, the buckling mode for the plastic cylinder is different from the corresponding elastic cylinder. The elastic cylinder at an internal pressure of 35 psi (241 kN/m<sup>2</sup>) buckles in a mode with  $m = 5$  and  $n = 6$ , whereas the plastic cylinder buckles in a mode with  $m = 5$  and  $n = 5$ , as shown by the lower plot in figure 3. At higher pressures, the modes differ by even more. For a pressure of 55 psi (379 kN/m<sup>2</sup>), for example, the elastic mode is again  $m = 5$  and  $n = 6$ , whereas the plastic mode is  $m = 4$  and  $n = 0$ .

The calculated in-plane stresses in the cylinder at buckling are given in figure 4. The axial load carried by the skin decreases as the internal pressure increases, whereas the load carried by the stringers remains relatively constant. Both the rings and the skin participate in carrying the hoop-tension load caused by internal pressure. The buckling load at an internal pressure of zero (fig. 3) is slightly greater ( $\approx 1.4$  percent) than the

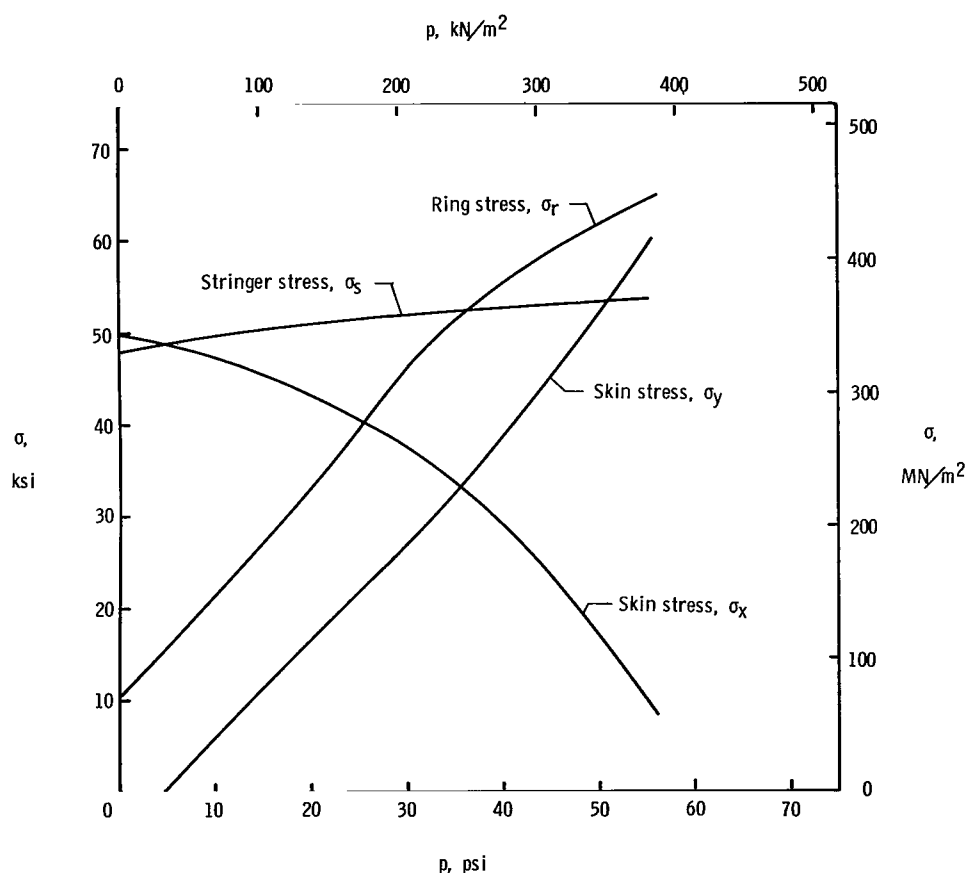


Figure 4.- Stresses in skin and stiffening members of ring-and-stringer-stiffened cylinder at buckling.  $\sigma_x$  and  $\sigma_s$  are compressive stresses;  $\sigma_y$  and  $\sigma_r$  are tensile stresses.

buckling load which would be computed with the use of reference 6. The use of equation (35) of reference 6 assumes that  $\sigma_x = \sigma_s$  and  $\sigma_y = \sigma_r = 0$  in contradistinction to the results shown in figure 4 and accounts for the small difference in predicted buckling load. The assumption of reference 6 tacitly presumes that prebuckling loading of the cylinder does not load the rings. Such a stress distribution might be obtained, for instance, with rings which have negligible stiffness prior to buckling but have their calculated stiffness during buckling. Such a device is normally used in classical buckling problems of cylinders in determining the prebuckling stress distribution in the cylinders resulting from restraint by the supports at the ends of the cylinders. In the present calculation, the rings have their calculated stiffness both prior to and during buckling, but the restraint offered by the rings is assumed to result in a stress distribution which does not vary with the coordinates  $x$  and  $y$ .

While the assumption  $\sigma_x = \sigma_s$  resulted in a negligible error for the special case of an axially loaded cylinder, sizable errors would result from the use of this assumption for a plate or cylinder under biaxial loading and the assumption should be replaced with the assumption used in the present example or some other suitable assumption.

Sandwich cylinder.- The sandwich cylinder analyzed herein is typical of those which might be selected for the intertank or interstage section of a large launch vehicle. The cylinder has face sheets of 7075-T6 aluminum alloy, a radius of 200 inches (5.08 m), a length of 200 inches (5.08 m), and an aluminum honeycomb core with a solidity of 2.5 per cent. The bonding material is assumed to have a mass of 0.15 psf (0.73 kg/m<sup>2</sup>). The face-sheet thickness  $t_f$  and depth of sandwich  $h$  are varied in order to arrive at desirable values of these dimensions for a sandwich cylinder subjected to a given load  $\bar{N}_x$ . For this purpose, the core shear moduli are taken to be  $G_{xz} = G_{yz} = 13.0$  ksi (89.6 MN/m<sup>2</sup>). This value of  $G_{xz}$  is roughly one-third of the shear modulus which would be obtained for the specified core from a conventional shear-stiffness test. The reduced value is used to account for the onset of plasticity in the core and for buckling of the core elements prior to buckling of the cylinder. A more elegant procedure to account for these effects is given in reference 14. Reference 14 also presents a method of accounting for that portion of the applied load carried by the core and bonding material. For simplicity, these effects are neglected in the present calculation.

Results of calculations made with the use of equation (D1) are given in figure 5 where the parameter  $\bar{N}_x/\bar{t}$  is plotted against the buckling load  $\bar{N}_x$  for various values of the face-sheet thickness  $t_f$ . Sandwich depth  $h$  varies along each of the curves. The parameter  $\bar{N}_x/\bar{t}$  is a measure of the efficiency of the sandwich cylinder in carrying the applied load. This parameter is normally chosen as large as possible in order to arrive at the wall configuration with the least mass.

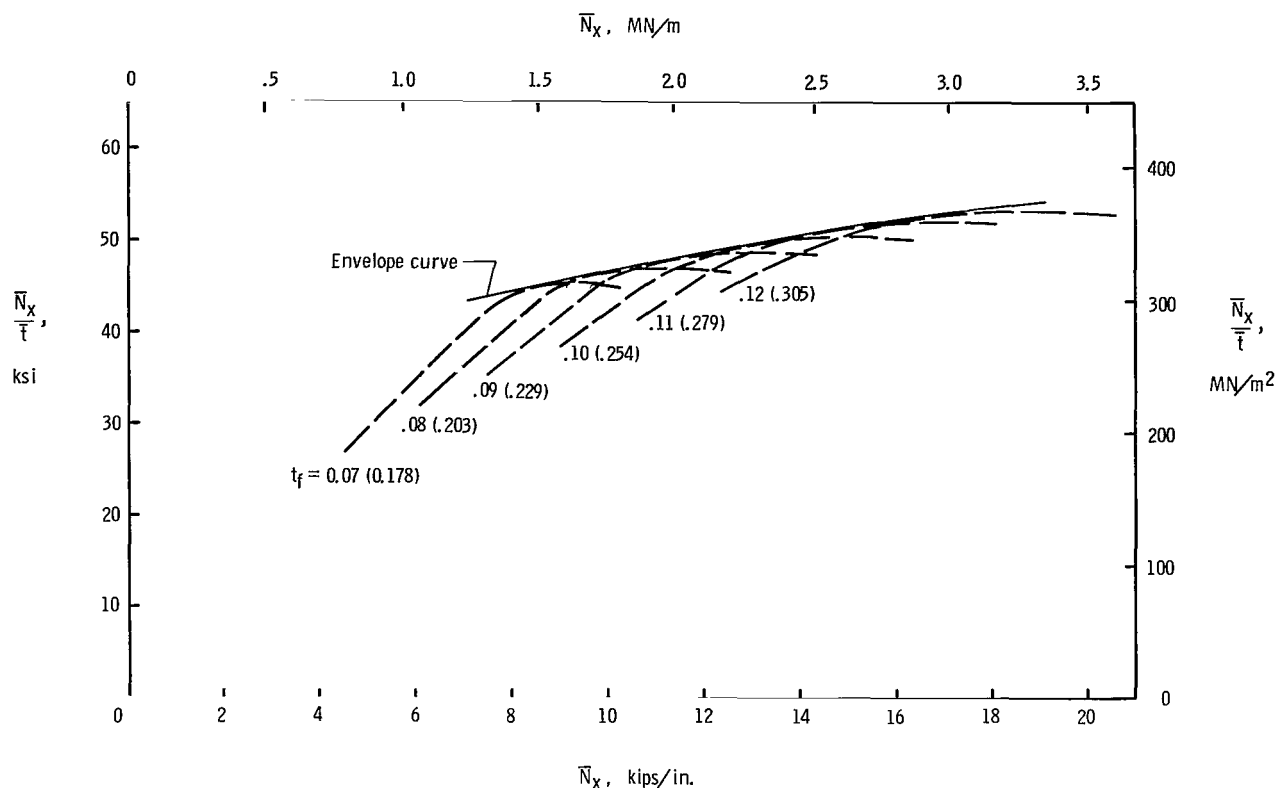


Figure 5.- Results of calculations to determine desirable proportions for sandwich cylinders under axial load. Linear dimensions are given in inches and parenthetically in centimeters.

For a load of  $\bar{N}_x = 11.0$  kips/in. (1.93 MN/m), the maximum value obtained in the previous numerical example, the highest attainable value of  $\bar{N}_x/\bar{t}$  is approximately 47.8 ksi (330 MN/m<sup>2</sup>). The corresponding sandwich proportions are a face-sheet thickness  $t_f$  of approximately 0.087 inch (2.21 mm), a value of  $\bar{t}$  of approximately 0.230 inch (5.84 mm), and a value of  $h/t_f$  of approximately 22.0. The stress in the face sheets at buckling ( $\approx 63.2$  ksi (436 MN/m<sup>2</sup>)) is well into the plastic region of the face-sheet material (see stress-strain curve of fig. 1) and is indicative of the need for considering plasticity in buckling calculations for low-mass sandwich structures. The use of simpler methods, which imply elastic behavior, would result in considerably different wall proportions. The methods of references 15 and 16, for instance, suggest that  $h/t_f$  should be approximately 80 and that approximately half of the mass of the sandwich wall should be allocated to the core material. The present calculation indicates that  $h/t_f$  should be about 22.0 and that only about one-fifth of the mass of the sandwich is in the core material.

The mode of buckling for the sandwich cylinder analyzed is the axisymmetric mode ( $n = 0$ ). For buckling in this mode, the buckling equation (eq. (D1)) simplifies

considerably; for moderately long cylinders with walls that experience moderate transverse shearing deformations during buckling ( $\bar{N}_x < D_{Qx}$ ), the buckling equation (ref. 14) can be written as

$$\bar{N}_x = \bar{N}_0 \left( 1 - \frac{\bar{N}_0}{4D_{Qx}} \right) \quad (13)$$

where

$$\bar{N}_0 = \sqrt{\frac{4E_y \bar{D}_x}{r^2}}$$

and where  $E_y$  and  $\bar{D}_x$  can be determined from equations (1) and (4). Use of equation (13) instead of equation (D1) in the given example would have resulted in predicted loads only slightly less than those given in figure 5. This difference results because equation (13) is not subject to the restriction that the buckle pattern must consist of a whole number of buckles.

**Ring-stiffened cylinder.**— Dimensions of the stiffening rings for the ring-stiffened cylinder analyzed are shown in figure 6. Other pertinent dimensions are wall thickness, 1.25 inches (3.17 cm); ring spacing, 20.0 inches (5.08 cm); radius, 200 inches (5.08 m); and cylinder length, 600 inches (15.24 m). The cylinder and rings are constructed of a steel with a compressive yield strength of 100 ksi (690 MN/m<sup>2</sup>). The stress-strain curve of the wall material is given in figure 1. The structure is representative of those which might be considered for the pressure hull of a submarine.

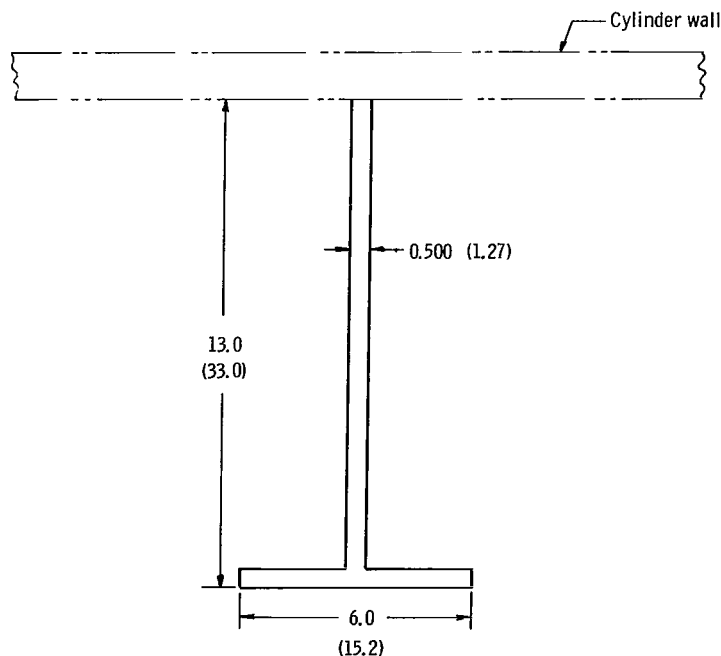


Figure 6.— Rings for ring-stiffened cylinder. All dimensions are given in inches and parenthetically in centimeters.

Results of the calculations are given in figure 7 in the form of an interaction curve. The results which apply for axial and circumferential load applied simultaneously and proportionately were obtained with the use of equation (15) of reference 13. The elastic calculations indicate a rather rectangular interaction curve. That is, circumferential load  $\bar{N}_y$  does not affect the axial load-carrying ability of the cylinder very much until  $\bar{N}_y$  is somewhat greater than  $2\bar{N}_x$ . At this point decreases in  $\bar{N}_x$  do not result in very substantial increases in  $\bar{N}_y$ .

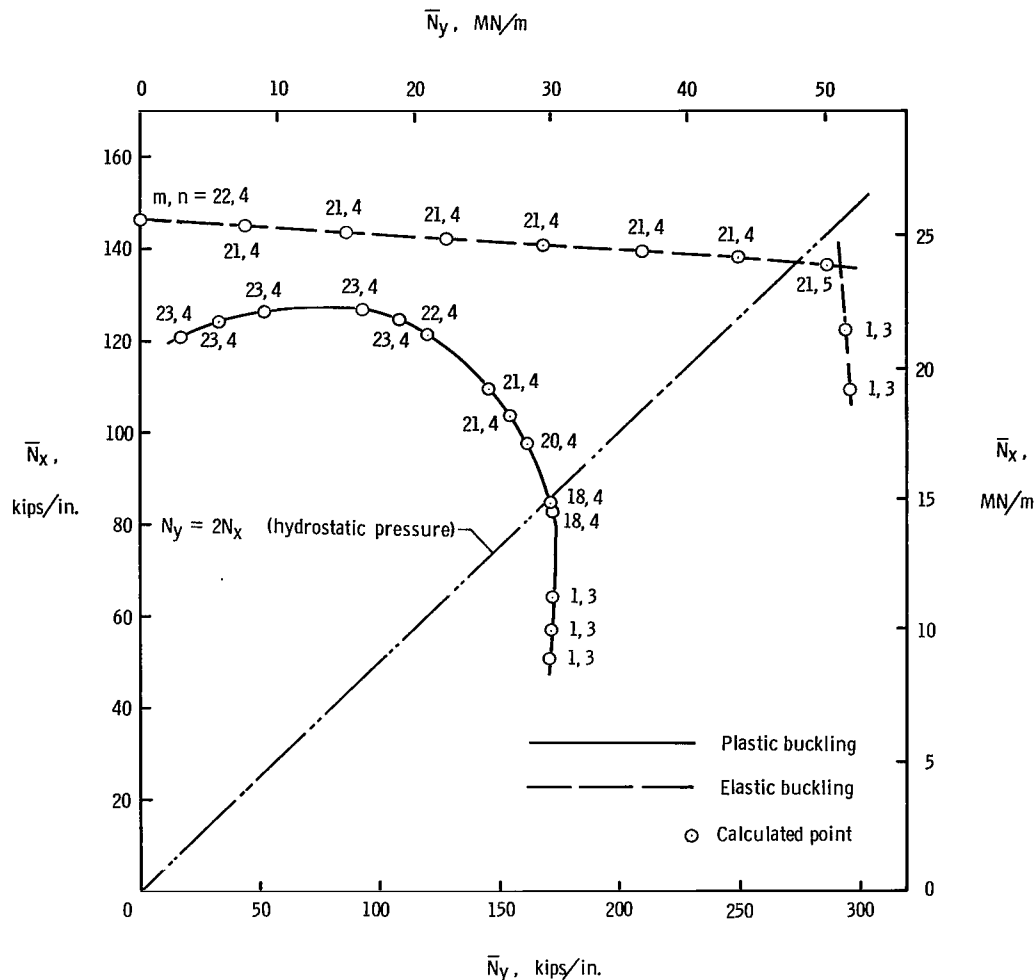


Figure 7.- Results of calculations to determine the buckling strength of a ring-stiffened cylinder under biaxial compressive loads.

A change in buckling mode is responsible for this behavior. For a small value of  $\bar{N}_y$ , the buckling mode consists of many axial buckles and eight or 10 circumferential buckles (four or five buckles on the semicircumference). For a large value of  $\bar{N}_y$ , the buckling mode consists of a single axial buckle and six circumferential buckles. Note that at  $\bar{N}_y = 2\bar{N}_x$  (hydrostatic pressure) the buckling mode consists of many axial buckles. Hence, use of buckling analyses (see, for instance, refs. 17 to 20) which assume that buckling occurs in a  $m = 1$  mode would result in an incorrect estimate of the buckling strength of the example cylinder which has a length-radius ratio of 3; for a shorter cylinder the error in predicted strength would be even more.

The interaction curve for plastic buckling is considerably more rounded in appearance than the curve for elastic buckling; the peculiar shape of the curve is a result of the

characteristic increase in performance of the plastic wall material under biaxial compression compared with that of the material under uniaxial compression. The reduction for plasticity in the hydrostatic case is large, with the plastic buckling load being approximately 62 percent of the elastic buckling load. No significant change in buckling mode is associated with the plastic behavior. The plastic cylinder buckles in a mode entailing many axial buckles (18 buckles instead of the 21 buckles in the elastic case) and eight circumferential buckles. For  $\bar{N}_y/\bar{N}_x$  somewhat greater than 2, buckling again is in a  $m = 1$  mode.

For buckling under hydrostatic pressure, the following stresses were obtained at buckling:

$$\sigma_x = 68.4 \text{ ksi } (472 \text{ MN/m}^2)$$

$$\sigma_y = 104.3 \text{ ksi } (719 \text{ MN/m}^2)$$

$$\sigma_r = 87.8 \text{ ksi } (605 \text{ MN/m}^2)$$

Note that the ring stress is only slightly above the proportional limit of the ring material, whereas the circumferential stress in the wall is greater than the compressive yield stress. Hence, the assumption that  $\sigma_y = \sigma_r$ , which is used in the application of many buckling analyses (see refs. 9 and 10, for instance), is not consistent with the stress-strain relationships used in the development of the analyses.

One additional calculation of interest was made in connection with the results shown in figure 7 which were obtained with the use of Donnell-type theory (ref. 13) that is known to be somewhat inaccurate for low values of  $n$ . In order to obtain some idea of the accuracy of Donnell-type theory in the present application, a check calculation assuming elastic behavior was made with the use of Sanders theory. The calculation was similar in all other respects to the elastic calculations of figure 7 in which buckling is indicated to occur in the mode with  $m = 1$  and  $n = 3$ . The check calculation indicated that Donnell-type theory predicted the correct mode shape ( $m = 1$  and  $n = 3$ ) but predicted a load that was approximately 6 percent too high for this section of the interaction curve of figure 7.

#### CONCLUDING REMARKS

A method of calculating the plastic buckling load of a wide variety of plates and shells with the use of available orthotropic buckling equations is presented and discussed. The method is restricted to plates and shells in which the biaxial stress prior to buckling is uniform. Despite this restriction, the method is applicable to many structures of current interest, many of which are not treated in the literature.



Plates and shells which may be readily analyzed by use of the proposed method may be of conventional, sandwich, or laminated construction. They may be reinforced with stiffening members either symmetrically disposed about the middle surface of the skin or eccentric to the middle surface; the stiffening members may be made of a different material than the skin. The effect of transverse shear stiffness on buckling may be taken into account in calculations for many structures.

The method of analysis applies to plates and shells subjected to any biaxial loading. Hence, pressurized cylinders under axial compression or cylinders subjected to axial compression and external pressure may be analyzed.

Results of three numerical examples are presented to indicate the types of structural problems which can be solved with the use of the method and to indicate the need for plastic buckling calculations in the design of contemporary structures. The results indicate that published methods for calculating the plastic buckling load of complex structures under hydrostatic loading are based on assumptions that are inconsistent with the stress distribution in the structure at buckling and which can result in significant errors in computed buckling load in some instances.

Langley Research Center,  
National Aeronautics and Space Administration,  
Langley Station, Hampton, Va., April 12, 1968,  
124-08-06-26-23.

## APPENDIX A

### CONVERSION OF U.S. CUSTOMARY UNITS TO SI UNITS

The International System of Units (SI) was adopted by the Eleventh General Conference on Weights and Measures in 1960. (See ref. 11.) Conversion factors for the units used in this report are given in the following table:

Physical quantity	U.S. Customary Unit	Conversion factor (*)	SI Unit (**)
Length . . . . .	in.	0.0254	meters (m)
Stress, modulus . . . . .	ksi	$6.895 \times 10^6$	newtons/meter <sup>2</sup> (N/m <sup>2</sup> )
Mass distribution . . . . .	psf	4.882	kilograms/meter <sup>2</sup> (kg/m <sup>2</sup> )
Pressure . . . . .	psi	$6.895 \times 10^3$	newtons/meter <sup>2</sup> (N/m <sup>2</sup> )
Load per unit length . . . .	kips/in.	$1.751 \times 10^5$	newtons/meter (N/m)

\*Multiply value given in U.S. Customary Unit by conversion factor to obtain equivalent value in SI Unit.

\*\*Prefixes to indicate multiple of units are as follows:

Prefix	Multiple
giga (G)	$10^9$
mega (M)	$10^6$
kilo (k)	$10^3$
deci (d)	$10^{-1}$
centi (c)	$10^{-2}$
milli (m)	$10^{-3}$

## APPENDIX B

### SYMMETRICALLY STIFFENED PLATES

The relationships given by equations (1) for stiffnesses of plastic plates of conventional, laminated, or sandwich construction can be readily modified to include the effect of stiffening which is symmetrical about the middle surface. The relationships could then be applied, for instance, to sandwich plates with truss-like or web-like cores, with the core elements being treated as stiffening elements such as stringers and rings or ribs. The relationships may also be used with some improvisation for problems entailing honeycomb cores (see ref. 14) if more precise calculations than would be obtained by neglecting the load-carrying capability of the core are desired.

The modified stiffnesses (denoted by \*) are

$$\left. \begin{aligned}
 \overline{E}_x^* &= \overline{E}_x + \frac{E_s A_s}{b_s} \\
 \overline{E}_y^* &= \overline{E}_y + \frac{E_r A_r}{b_r} \\
 G_{xy}^* &= G_{xy} \\
 \overline{D}_x^* &= \overline{D}_x + \frac{E_s I_s}{b_s} \\
 \overline{D}_y^* &= \overline{D}_y + \frac{E_r I_r}{b_r} \\
 D_{xy}^* &= D_{xy} + \frac{G_s J_s}{2b_s} + \frac{G_r J_r}{2b_r} \\
 \mu_x'^* &= \frac{\mu_x' \overline{E}_y}{\overline{E}_y^*} \\
 \mu_y'^* &= \frac{\mu_y' \overline{E}_x}{\overline{E}_x^*} \\
 \mu_x^* &= \frac{\mu_x \overline{D}_y}{\overline{D}_y^*} \\
 \mu_y^* &= \frac{\mu_y \overline{D}_x}{\overline{D}_x^*}
 \end{aligned} \right\} \quad (B1)$$

## APPENDIX B

where the plate terms without an asterisk are those given in the body of the paper (eqs. (1), (2), (4), and (5)) for conventional, laminated, or sandwich plates and must be evaluated at the stress intensity of the plate. The moments of inertia given for stringers and rings or ribs are those taken about the centroid of the symmetrical plate. The extensional modulus for a stringer or a ring (rib) is the tangent modulus of the member at the strain experienced by the member, whereas the shear modulus is a secant-type modulus  $\left(G_{\text{sec}} = \frac{E_{\text{sec}}}{2(1 + \mu)}\right)$  evaluated at the same strain.

Equations (B1) are similar to those given by equations (18) of reference 10 for the special case of a web-core sandwich plate except that the referenced equations make no distinction between the stress in the face sheets and the stress in the core of the sandwich plate. Equations (B1) also reduce to those of reference 9 for the special case of a ring- and/or stringer-stiffened cylinder of conventional construction when the stiffening is symmetrical about the middle surface of the skin. In reference 9, as in reference 10, no distinction is made between the stress in the skin and the stress in a stiffening element.

## APPENDIX C

### TRANSVERSE SHEAR STIFFNESS OF TRUSS-CORE AND WEB-CORE SANDWICH PLATES

In order to apply the equations given earlier to sandwich plates and shells with web-like or truss-like cores, the transverse shear stiffness of the core must be known. Calculations are normally made by using approximate expressions for these stiffnesses such as those given in reference 21 for the elastic value of the stiffnesses; these expressions can be modified to account for the effects of plasticity. Such expressions are given herein.

The stiffness in a direction parallel to the direction of the core elements for both web-like and truss-like cores is given by (ref. 21)

$$D_Q = \frac{E_c}{2(1 + \mu)} \beta \sin^2 \varphi \frac{h^2}{h - t_f} \quad (C1)$$

where  $E_c$  is the secant modulus of the core material evaluated at the uniaxial stress (strain) experienced by the core and  $\beta$  is solidity of the core, that is, the ratio of volume occupied by core elements to the total volume available between the face sheets of the sandwich. Equation (C1) was derived on the assumption that the core elements acted like shear webs to inhibit shear displacements of the face sheets; hence, the secant modulus appears in equation (C1) to account for plasticity. For a web-core sandwich plate,  $\varphi$  is equal to  $90^\circ$  and equation (C1) is equivalent to equation (48) of reference 10.

For truss-core sandwich plates, the stiffness perpendicular to the direction of the core elements is given by

$$D_Q = \bar{E}_c \sin^2 \varphi \cos \varphi \quad (C2)$$

for a single truss and by

$$D_Q = 2\bar{E}_c \sin^2 \varphi \cos \varphi \quad (C3)$$

for a double truss. For  $\mu = \mu_e = \frac{1}{2}$ , the quantity  $\bar{E}_c$  becomes

$$\bar{E}_c = \frac{E_{\text{sec}^t c}}{1 - \mu^2} \quad (C4)$$

where the secant modulus of the core material is evaluated at the unidirectional stress (strain) experienced by the core elements. In the derivation of equations (C2) and (C3),

## APPENDIX C

a shear-like deformation is assumed to result from plate stretching of the core elements while the core is subjected to the applied uniaxial stress.

The expression for the stiffness perpendicular to the direction of the core elements for a web-core sandwich is complicated by the fact that bending deformations of both the face sheets and the core elements are considered to contribute to the shear-like deformation; only deformations of the core elements were considered in the previous derivations. The stiffness is given by

$$D_Q = \frac{1}{\frac{b_s h}{12\bar{D}_c} + \frac{b_s^2}{24\bar{D}_f}} \quad (C5)$$

where  $\bar{D}_c$  refers to the bending stiffness of the core element and is given by

$$\bar{D}_c = \frac{E_{\text{sec}} t_c^3}{12(1 - \mu^2)} \quad (C6)$$

The secant modulus of the core material is again evaluated at the unidirectional stress (strain) experienced by the core elements. The bending stiffness  $\bar{D}_f$  is given by  $\bar{D}_x$  or  $\bar{D}_y$  of equations (1) depending upon whether the core elements are aligned in the y-direction or in the x-direction, respectively. Equation (C5) was adapted from reference 22, the source of the corresponding equation of reference 21, and is the equivalent of equation (37) of reference 10.

## APPENDIX D

### BUCKLING OF SIMPLY SUPPORTED ORTHOTROPIC PLATES AND CYLINDERS

The availability of a buckling equation for orthotropic plates and cylinders in which the shear stiffnesses  $D_{Q_x}$  and  $D_{Q_y}$  are distinct quantities is desirable for making plastic buckling calculations involving sandwich-like structures. Such an equation is readily written for simply supported plates and cylinders subjected to biaxial in-plane loads with the use of reference 12 as

$$\bar{N}_x \left( \frac{m\pi}{l} \right)^2 + \bar{N}_y \left( \frac{n\pi}{b} \right)^2 = A_{11} - \frac{A_{22}A_{13}^2 + A_{33}A_{12}^2 - 2A_{12}A_{13}A_{23}}{A_{22}A_{33} - A_{23}^2} \quad (D1)$$

where

$$A_{11} = \bar{D}_x \left( \frac{m\pi}{l} \right)^4 + D_{12} \left( \frac{m\pi}{l} \right)^2 \left( \frac{n\pi}{b} \right)^2 + \bar{D}_y \left( \frac{n\pi}{b} \right)^4 + \frac{E_x E_y}{r^2} \frac{\left( \frac{m\pi}{l} \right)^4}{E_x \left( \frac{m\pi}{l} \right)^4 + E_{12} \left( \frac{m\pi}{l} \right)^2 \left( \frac{n\pi}{b} \right)^2 + E_y \left( \frac{n\pi}{b} \right)^4}$$

$$A_{12} = \bar{D}_y \left( \frac{n\pi}{b} \right)^3 + \frac{D_{12}}{2} \left( \frac{m\pi}{l} \right)^2 \left( \frac{n\pi}{b} \right)$$

$$A_{13} = \bar{D}_x \left( \frac{m\pi}{l} \right)^3 + \frac{D_{12}}{2} \left( \frac{m\pi}{l} \right) \left( \frac{n\pi}{b} \right)^2$$

$$A_{22} = D_{Q_y} + \bar{D}_y \left( \frac{n\pi}{b} \right)^2 + \frac{D_{xy}}{2} \left( \frac{m\pi}{l} \right)^2$$

$$A_{23} = \left( \frac{D_{12} - D_{xy}}{2} \right) \left( \frac{m\pi}{l} \right) \left( \frac{n\pi}{b} \right)$$

$$A_{33} = D_{Q_x} + \bar{D}_x \left( \frac{m\pi}{l} \right)^2 + \frac{D_{xy}}{2} \left( \frac{n\pi}{b} \right)^2$$

and

$$D_{12} = 2D_{xy} + \mu_x \bar{D}_y + \mu_y \bar{D}_x$$

$$E_{12} = \frac{E_x E_y}{G_{xy}} - \mu'_x E_y - \mu'_y E_x$$

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Equation (D1) was obtained by substituting the equations

$$\left. \begin{aligned} w &= A \sin \frac{m\pi x}{l} \sin \frac{n\pi y}{b} \\ Q_y &= B \sin \frac{m\pi x}{l} \cos \frac{n\pi y}{b} \\ Q_x &= C \cos \frac{m\pi x}{l} \sin \frac{n\pi y}{b} \end{aligned} \right\} \quad (D2)$$

into equations (7), (8), and (14) of reference 12 which are based on a small-deflection theory for buckling in which membrane prebuckling deformations of the plate or cylinder are assumed. The implied boundary conditions at a boundary  $x = 0$  or  $x = l$  are  $w = M_x = v = N_x = Q_y = 0$  (refs. 12 and 23) and at a boundary  $y = 0$  or  $y = b$  are  $w = M_y = u = N_y = Q_x = 0$ . The buckling load of a plate or cylinder is obtained from equation (D1) by minimizing with respect to  $m$  and  $n$ , the number of half waves in the buckle pattern in the length and width directions of the plate, respectively. The smallest  $n$  consistent with the assumption of simply supported plates is  $n = 1$ . When the equation is applied to a cylinder, it is convenient to take  $b = \pi r$  and  $n = 0, 1, 2, 3$ , and so forth. If buckling is determined to occur in the  $n = 1$  mode or the  $n = 2$  mode, which corresponds to a lateral displacement of the cylindrical cross section and to a flattening of the cylindrical section to an oval section, respectively, a more exact buckling theory should be used. Reference 12 is based on Donnell-type theory and buckling loads obtained for the  $n = 1$  and  $n = 2$  modes are not very accurately given by the theory for certain loading conditions (ref. 23). For both plates and cylinders  $m$  is taken to be  $m = 1, 2, 3$ , and so forth.

Equation (D1) is used in the body of the paper to compute plasticity factors and to compute buckling loads for plates and cylinders which buckle plastically.



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